Faults and Mathematical Disagreement

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Abstract: My aim in this paper is to analyse the notion of mathematical disagreements and, particularly, the possibility of there being “faultless disagreements” in mathematics. In order to do so, and after having clarified the basic concepts involved in the debate, I compare disagreements concerning basic axioms in mathematics (i.e. the axiom of choice) with aesthetic disagreements. I conclude that there cannot be faultless disagreements in mathematics, either they are mere misunderstandings or one of the agents involved is mistaken (at fault). In this brief notes, I elaborate on an objection raised against this conclusion involving the use of intuitionist logics and I argue that, because of the requirement of semantic competence, it is not conclusive.

Key words: Disagreement; faults; cognitively ideal agents; mathematical objectivity; semantic competence.

Introduction

Mathematics is usually seen as the most objective of the human enterprises. It is popularly conceived as a unified and stable body of universal truths (universally agreed by all and universally applicable to all). It doesn’t take much to dismantle this popular view however. It suffices to take a superficial look at the debates over the foundations of mathematics to see that taking a stand on some of these is not a black and white issue. It is hardly controversial to

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claim that when it comes to foundational issues in mathematics, discussions and controversies seem to turn intractable. Indeed, some of these disagreements, the ones we will be concerned with here, share one common characteristic: they are persistent.¹

I believe many of the philosophical proposals concerning mathematics can be seen as attempts to give an explanation of this persistent character of mathematical disagreements. Most of these attempts though focus on ways to determine the truth conditions of mathematical statements, the nature and ontological status of mathematical entities or, more generally, what criteria there are to decide what axioms should be accepted and which not. I propose to take a step backwards somehow and instead of questioning directly what it is that should determine who is right and who is wrong in mathematical disagreements (that is, what are the truth conditions or the criteria to determine the truth value of mathematical statements), I propose that we take some time to analyse the very nature of mathematical disagreements. I do not intend to suggest of course that the, let’s say, “traditional” approaches are wrong or not interesting. Quite the opposite. But I believe much can be learned from the perspective proposed here.

I believe that focusing on the semantic analysis of mathematical disagreement will give us, first, a new and ontologically-free way to approach the topic. Second, it will enable us to draw analogies with other areas of discourse, analogies that, I claim, will shed some surprising results. And, finally, from such a neutral analysis, some important consequences can be extracted, both ontological and epistemological and, also and importantly, some consequences for the role of philosophy in mathematical practice.

Now, I will not attempt to give a full account of my arguments in this short notes; rather I will focus on some problems and objections that were raised concerning some basic definitions and I will attempt to give a proper answer to them. By doing so, I believe, I will improve substantially my initial approach and I

¹ I’m not saying of course that this is the only feature they share, only that this is the one I want to focus on.
thus I will reinforce one of the conclusions of my paper: that there cannot be faultless disagreements when it comes to disagreements about the validity of foundational axioms in mathematics.

Two examples

Let’s begin presenting examples of two, in principle, very different situations. Consider first the case of two mathematicians\(^2\) who meet at a colloquium (say, at one meeting of the philosophy of mathematical practice association) and start arguing hotly about the validity of the axiom of choice (AC, from now on):

1. Ana: The Axiom of Choice is a valid axiom and it should definitely accepted.

2. Jon: No you are wrong, it is not only invalid, but also unnecessary.

Let’s assume they both know the history of the debate over the AC, all the arguments for and against it and still, as in fact it happens, they cannot reach an agreement. The problem thus seems to be intractable. One could even conclude that there is just no solution to it, but why is that? Isn’t mathematics after all, an objective and rational subject? Wouldn’t that objectivity entail that one of them has to be wrong, even if it is impossible to decide who?, or is it possible that both are right?

Consider this other case. Imagine a situation where two very rich and well-educated friends, say Joan and Mariano, are at an auction with the intention of purchasing a nice picture to decorate their shared mansion, but they are having trouble in choosing it, thus the following exchange occurs,

\(^2\) Or two philosophers of mathematics. From now on we will talk about mathematicians, if only because it is shorter, but you can substitute it for a philosopher, without, I want to believe, any substantial loss.

3. Joan: William Turner is much better than Monet. We should definitely get one of his.

4. Mariano: No, I will not have it! Monet is much better than Turner!

Now, in this case, the lack of agreement is not only less shocking, it would seem even normal. After all, tastes are a subjective issue, and it could be argued that Joan and Mariano are merely exchanging their opinions, sharing their standards of taste and beauty. But then again they are clearly disagreeing. For one, Mariano’s statement is directly contradicting Joan’s; wouldn’t this imply that he thinks she is wrong? Not only that her idea of beauty or what it means to be better when it comes to art is wrong, but rather that she is wrong, that there is something she is missing (i.e. she doesn’t understand enough of art, that Monet is more expensive and thus better, that according to a number of experts Monet is better, and thus she is wrong, etc.).

These two exchanges, in principle quite different, share some important elements. Sadly perhaps (at least for mathematicians), the clearest of them is that both disagreements seem intractable. This is probably due to the fact that in both cases it is difficult to know what exactly it is that they are disagreeing about and, even worse, whenever we postulate a candidate, we encounter epistemological and ontological problems to ensure that this is so. This results in the impression that, in both cases, none of the people involved is committing any mistake, that is, none is at fault. Still, there are standards out there that should or at least could help us decide on these issues: in the first case, mathematical and logical standards; in the second, cultural or community based standards. But it is not clear, on the one hand, what precisely are these standards and, on the other hand, whether they are sufficient to determine who is right and who is wrong in these exchanges. For, in order to claim that these are cases of disagreements, it seems that we have to assume that at least one of the discussants is wrong. Or do we?
Introducing the Concepts

Disagreement and Fault

First it is important to establish what we mean by disagreement. It is particularly important to distinguish what I shall call genuine disagreement from weaker cases, that is, cases involving ambiguity or misunderstanding. This is trickier than it might seem at first, and it has proven to be a particularly problematic task. Indeed, I shall offer a definition of disagreement but, as we shall see it will prove to be insufficient, in the best of the cases, and, in others, it will just lead us to erroneous conclusions. How to enrich it, will be the main issue I will try to address here.

According to the definition I proposed, Genuine Disagreement happens whenever A being right in saying $P$ entails that B is wrong in saying $not-P$. More precisely, for any parameter $i$ truth depends on, $P$ is true with respect to $i$ and $not-P$ is false with respect to $i$.

Now, the first thing that might strike as problematic here is that I seem to be offering two definitions, and not just one. I’d like to claim here that this is not only not a problem, but that it is also necessary. Further, I will end up postulating yet another requirement for a disagreement to count as a genuine one: the requirement of the speakers being semantically competent.

But before doing so, let me finish introducing the key concepts in the debate. When both Joan and Mariano and Jon and Ana were disagreeing I claimed that one of the common elements in their exchanges was that none of them seemed to be committing any mistake; that is, none of them were at fault. A quick way to understand what I mean by “fault” is through the introduction of the idea of cognitively ideal agents, i.e. agents not committing any cognitive fault and not suffering from any cognitive limitation, be it logical outsight, relevant
ignorance, insincerity, unimaginativeness, etc.\textsuperscript{3} Now if, after assuming that the speakers involved in the controversy are cognitively ideal subjects, the disagreement remains unsolved (and we have proven it is genuine) then we’ll have to conclude that it is a case of faultless disagreement: disagreements were A claims that $P$ and B claims that not-$P$ but neither A nor B are at fault\textsuperscript{4}.

Now, I want to claim that there cannot be FD in mathematics, that either they are mere misunderstandings (or other possible cases of weak disagreement) or one of the speakers (or both!) is committing some sort of mistake. This leads me to defend some further claims, mostly involving the nature of mathematical objectivity and the role played by mathematical practice and mathematical community in the establishing of the standards to decide upon the validity of certain axioms. But, relevant as I believe these are for my overall view of mathematical truth and mathematical objectivity, I will not discuss them here. Rather, as I said, I will focus on one particular objection raised against the very basic definitions I presented of both disagreement and fault.

\textit{Disagreement, AC and Intuitionist Logic}

Recall that the definition of genuine disagreement presented had two parts to it. Now, I will try to argue for the need of both, indeed, I will end up adding yet another part or requirement to the definition.

The first part of the definition sates: “$A$ being right in saying $P$ entails that $B$ is wrong in saying not-$P$”. As it has been pointed out (Marco Panza, private communication), suppose now that $A$ endorses AC and $B$ endorses not-AC. Suppose also that $A$ and $B$, together with an external evaluator of their practice, accept that the good logic for doing mathematics is not classic but rather, say, standard intuitionist logic.

\textsuperscript{3} I introduce the idea of cognitively ideal agents to avoid confusions with epistemic fault, that is, cases where there is an objective element to determine the truth-value of the assertions in play, but both speakers are ignorant of it. Here I will be concerned with cases in which there is just no such objective element and if there is, it would be impossible for a cognitively ideal agent to know it.

\textsuperscript{4} The literature on faultless disagreements is huge; see Köbel (2003) for a nice introduction to it.

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\textit{Notae Philosophicae Scientiae Formalis,} \\
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In this logic, it is still the case that $AC$ being true entails not-$AC$ being false. But neither $AC$ nor not-$AC$ can be said to be true or false. Then, it seems, we find that both $A$ and $B$ are genuinely disagreeing but, nonetheless, none of them is at fault! That is, we would be facing a clear case of FD.

And here is where the second element of our definition comes into play: “for any parameter $i$ truth depends on, $P$ is true with respect to $i$ and not-$P$ is false with respect to $i$”. That is, $A$ and $B$’s assertions, or Jon and Ana’s assertions, in our example, have to be evaluated with respect to one and the same parameter. What this parameter might be, in the case of the $AC$ as in the case of Monet/Turner, is a complex issue. Indeed, it can be said that it is the issue. But whatever its nature is: standards determined by mathematical practice, the very nature of sets or whatever other possibility that has been brought up in the literature; the fact remains that it has to be common for both speakers.

Now, if we are working within the framework of classical logic, being bivalent, either $AC$ or not-$AC$ must be true and thus we would have to conclude that, if they are disagreeing then either Jon or Ana must be at fault. But assuming an intuitionist frameworks will not alter this conclusion for, if indeed neither $AC$ nor not-$AC$ can be said to be true or false, then that implies that there is not a single parameter $i$ with respect to which $AC$’s truth value is to be evaluated. Or, to be precise, that we cannot know that parameter, we do not have access to it. But then, I want to claim, Jon and Ana will not be genuinely disagreeing.

Defending that they are genuinely disagreeing would entail defending that they are semantically blind and so they are ignorant as to what circumstances of evaluation are relevant for the assessment of their utterances. But that implies that they are ignorant of the truth-conditions of their utterances and of their meaning. And that is certainly not a welcoming perspective. And it is even less welcomed if we remember that we are considering here cognitively ideal agents.

Among other things, semantic blindness will erase the intuition that there is a disagreement all together. If we accept the idea that the speakers do not
know the meaning of their utterances, we can hardly claim that they are disagreeing. The most we could say is that they are talking pass each other.

It seems that a basic requirement for an exchange to count as a disagreement is that the disputants know the meaning of their utterances. It also seems important, for that matter, that the utterances are assessed according to the speakers’ intentions\(^5\). In other words, the speakers have to be semantically competent\(^6\).

So, I think the requirement of semantic competence needs to be included in the definition of genuine disagreement given above. The resulting, enriched definition would thus be something like:

\[ \text{Genuine Disagreement, according to which A being right in saying } P \text{ entails that B is wrong in saying not-} P \text{ More precisely, for any parameter } i \text{ truth depends on, } P \text{ is true with respect to } i \text{ and not-} P \text{ is false with respect to } i. \text{ And both A and B need to be semantically competent.} \]

**Some Concluding Remarks**

The purpose of these short notes was to discuss further some of the objections raised against my definition of disagreement. Needless to say, much work remains to be done. I do not intend to claim that this is a final definition, some other problems need to be solved before. The idea of semantic competence, and its relevance in the case of mathematical axioms, needs to be developed further. The idea of a parameter \( i \) is a key notion here and,  

\(^5\) This is an important point. Determining the truth-conditions of assertions is a complex issue and the intentions of the speakers play a fundamental role in it. It is the speakers who determine for instance, the framework within which they are talking (i.e. a particular set-theory or logic, thus with different rules to determine the truth-value). Through their utterances, they express their intentions, and it is essential for communication that the hearer captures them correctly. If something goes wrong in this process, we would be facing, once again, a case of misunderstanding, and not of genuine disagreement. See Korta and Perry (2011) for an elaborate theory on the role of intentions in communication and truth determination.

\(^6\) Stojanovic (2008) also mentions Semantic Competence as a requirement for disagreement and offers a definition of it for predicates of personal taste: “Speakers of English are semantically competent with predicates of taste: they master their meaning and truth conditions”.

admittedly, a very controversial one. Also, I am aware of some unfortunate inaccuracies in my terms. For example, there is a difference between saying that the AC is valid and saying that it is true. An axiom can be accepted without its truth having been proved (and hence there could be genuine disagreement about the convenience to adopt it that do not imply disagreeing about any parameter to determine its truth value).

In a longer version of these notes I attempt to address these potential inaccuracies, I try to explain them and justify them in light of further considerations regarding the role of mathematical practice in securing mathematical objectivity. But developing a proper definition of disagreement is a key aspect of my project. And this is what I intended to do in these short notes.

References