Husserlian Sets or Fregean Sets?

Claire Ortiz Hill
Centre National de la Recherche Scientifique
claire.hill@orange.fr

Abstract: Few realize that Edmund Husserl elaborated theories about sets and the causes of the set-theoretical paradoxes. Here I examine his convictions: that his contemporaries did not yet have the real and genuine concept of set needed; that if one is clear and distinct with respect to meaning, one readily sees the contradiction involved in the set-theoretical paradoxes; that the solution to them would then lie in demonstrating the shift of meaning that makes it that one is not immediately aware of the contradiction and that once one perceives it one cannot indicate wherein it lies. I study these convictions in connection with the conclusions that he, Frege and Russell came to regarding the causes of the paradoxes derivable within Frege's system.
Key words: sets; set-theoretical paradoxes; Husserl; Frege.

Here I seek to provide the conceptual framework for interpreting Husserl's statements that: 1) the set-theoretical paradoxes show that his contemporaries did not have the concept of set needed; 2) if one is clear about meaning, one sees the contradiction involved in those paradoxes; 3) the solution to them lies in demonstrating the shift of meaning that makes one not immediately aware of the contradiction and unable to indicate wherein it lies. I compare issues involved in Frege's use of the extensions that lead to the contradiction about the set of all sets that are not members of themselves and conclusions he and Russell came to regarding its causes.

By the late 1890s, Husserl considered it very important to distinguish between logical laws and laws of meaning. For him, logical laws guarded against formal or analytical contradiction, *Widersinn*. What violates them, what is contradictory, *widersinnig,* has a coherent meaning, can be true or false, but no existing object can correspond to the meaning. As an example of a *Widersinnigkeit*, he gave, 'The present emperor of France is blond.' Similarly, he held that statements about sets containing themselves as an element are *widersinnig*.

In contrast, laws of meaning distinguish meaningfulness from meaninglessness, sense from nonsense, by providing pure logic with possible coherent, meaningful meaning forms whose truth or falsehood and reference to objects, logical laws determine. Meanings, Husserl stressed, are governed by *a priori* laws that regulate how they may be combined and constitute meaningful, coherent meanings instead of nonsense (*Unsinn*). The impossibility of combining meanings in certain ways is not subjective, but objective and grounded in the pure essence of meaning.

Husserl believed that the primitive, essential distinction between dependent and independent meanings formed the necessary basis for discovering the essential categories of meaning in which essential laws of meaning are grounded. Like Frege before him and Russell after him, Husserl stressed that differences between dependent and independent meanings concealed behind inconspicuous grammatical distinctions are inviolable because "founded deep in the nature of things." He studied how one is led astray by the fact that, though meanings of each category figure in the subject position, their meanings are then no longer the same. Not just any meaning can be substituted for *S* or for *p*. Once meaning categories are violated, coherency of meaning is lost.

The underlining on Husserl's copies of Frege's "Concept and Object" and "Function and Concept" shows Husserl's agreement with Frege here.

Husserl also detected a natural order in formal logic and broadened its domain to include two levels above Aristotelian logic, which he considered a small area of pure logic needing to be segregated from the extended sphere of pure logic that embraces the mathematical disciplines and is immense in comparison. He considered his understanding of the structure of the world of pure logic a radical clarification of the relationship between formal logic and formal mathematics and that it led to a definitive clarification of the sense of pure formal mathematics as a pure analytics of non-contradiction.

On Husserl's first level, the Aristotelian logic of subject and predicate propositions and states of affairs deals with what is stated about objects in general from a possible perspective. On the two higher levels, it is no longer a question of objects as such about which one might predicate something, but of investigating what is valid for higher-order objective constructions that are determined in purely formal terms and deal with objects in indeterminate, general ways. On the second level, Husserl located set theory, the basic concepts of mathematics, mathematical physics, formal pure logic, pure geometry, pure theory of meaning, the logic of morality, the ontology of ethical personalities, the pure logic of values, pure esthetics, the logic of the ideal state, the ideal of a valuable existence, etc. His third level is that of his theory of manifolds. We shall not be concerned with it here.

The key thing to realize here is that Husserl held that sets and numbers function in an entirely different way on the first level than in set theory and arithmetic on the second level. In first-level expressions like '3 houses,' numbers occur as form, but not as independent objects about which something is predicated. In that case, the sentence 'Jupiter has four moons,' to use Frege's example, is a statement about Jupiter's moons in which four occurs as form and is thereby dependent. It is one thing, Husserl stressed, to make statements about objects in which numbers occur as dependent forms and another to make statements about numbers in which the numbers are objects. When we make such forms independent, new higher-order objects, hypostasizations of forms, emerge that are not objects in their own right. This is why numbers function entirely differently on the first level than in arithmetic on the second level, where statements in which numbers are objects are found, for example, 'Any number can be added to any number.'

Husserl considered the second level to be an expanded, completely developed analytics where one calculates, reasons deductively, with concepts and propositions. One proceeds in a purely formal manner since every concept is analytic. One manipulates signs that acquire meaning through the rules of the game. Signs and rules of calculation suffice because each procedure is purely logical. One proceeds mechanically and the result is accurate and justified.

From the late 1890s on, Husserl held that the world of the mathematical and purely logical was a world of concepts, where truth is nothing other than analysis of essences or concepts, and pure logical, mathematical laws are laws of essence. He defended the view, which he attributed to Frege's teacher Lotze, that pure arithmetic is a branch of logic. The unending profusion of theories that arithmetic develops is enfolded in its axioms, propositions. Following systematic, simple procedures, deduction unfolds the idea of cardinal number from some side, or unfolds ideas inseparably connected with it.

How do Husserl's ideas about sets and the set-theoretical paradoxes fit into this conceptual framework? First, remember that sets have an entirely different meaning in the subject-predicate propositions of Husserl's first level than in the set theory of his second level. On the first level, individual objects are the terms of the predication. Sets, however, do not occur as objects in subject-predicate propositions, but as dependent forms.

In the set theory of the second level, we do not make judgments directly about elements, but about whole totalities of arbitrary elements. The sets are the objects-about-which, as in the statement, '2 sets can each be joined into a new set.' Set theory is derived analytically from the concept of set, from a set essence, that is expressed in the relation between a set itself and its elements and makes it impossible for the members of the relation to be identical. It belongs essentially to the concept of set that no set can contain itself as an element without contradiction. In 1891, Husserl wrote regarding Schröder's attempt to show that bringing all possible objects of thought into a class gives rise to contradictions that in the calculus of sets, a set ceases to have the status of a set when as it is considered as an element of another set; and this latter in turn has the status of a set only in relation to its primary, authentic elements, but not in relation to whatever elements *of* those elements there may be." He warned that errors in inference result if one fails keep this in mind.

For Husserl, it is part of the idea of set to be a unit, a whole, comprising certain members as parts in such a way that it is something new that is first formed by them. It belongs essentially to the concept of whole that no whole can contain itself as a part. So, as a kind of whole, a set is subject to the rules governing wholes and parts that stipulate that a whole cannot, without contradiction, be its own part. Sets are *a priori* different from their members. Husserl repeatedly called the set theoretical paradoxes *Widersinnigkeiten*. For him, the set of all sets that are not members of themselves is a *Widersinnigkeit* that, like the present emperor of France, cannot be something that exists.

Husserl, Frege and Russell came to many of the same conclusions about the causes the set-theoretical paradoxes, so let us look at the reasoning that led Frege to introduce sets and Russell's struggles to evade the contradiction derivable in Frege's system.

Frege thought that we make a mistake that can easily vitiate our thinking if we fail to attach a reference to proper names and concept-words. He considered the prime problem of arithmetic to be that of how one apprehends logical objects, in particular numbers. Reasoning only on Husserl's first level, he argued that numbers were independent objects that must always be conceived substantivally and not as dependent attributes. He believed that the definite article in an expression like 'the number 4' classes it as an object and that in arithmetic this independence comes out at every turn, as in the identity 4 + 4 = 8. He thought that the fact that numbers also appear attributively could "always be got around." For example 'Jupiter has four moons' could be rewritten as 'the number of Jupiter's moons is four,' which is an identity stating that the expression 'the number of Jupiter's moons' signifies the same object as the word 'four.' The independence he claimed for numbers was to preclude using such words as predicates or attributes, which considerably alters their meaning.

Recognizing that his formula for treating what is dependent as independent leads to evidently false, nonsensical or sterile conclusions, Frege settled for the definition: "The Number which belongs to the concept F is the extension of the concept 'concept equal to the concept F'" and for his axiom of extensionality, which he considered unprovable. Upon learning of Russell's contradiction, he pronounced the law false. He confessed he had been reluctant to use classes, but saw no other way to apprehend logical objects.

Frege described the shift of meaning that had made him not immediately aware of the contradiction. The paradoxes of set theory arise, he said, because a concept is connected with something called a set which appears to be determined by the concept and determined as an object. Such a transformation is inadmissible, because there is no such object. He described the "essence of the procedure which leads to the thicket of contradictions." The objects falling under *F* are regarded as a whole, as an object and designated by the name 'set of *F*s.' This is inadmissible because of the essential difference between concept and object, which is covered up in language. Confusion arises if, as a result of its transformation into a proper name, a concept-word is in a place for which it is unsuited.

Russell said that his struggle with the contradiction taught him that if a word or a phrase that is devoid of meaning is wrongly assumed to have an independent meaning, false abstractions, pseudo-objects, paradoxes and contradictions may result. He had believed that, when it is said that a number of objects all have a certain property, it is supposed that the property is a definite object that can be considered apart from the objects having the property. It is also supposed that the objects having the property form a class, a new single entity, distinct from each member of the class.

However, the contradiction showed him that classes are radically different from individuals. He reasoned that if you think that classes are things in the sense in which things are things, you will have to say that the class of all the things in the world is a thing in the world and therefore a member of itself. He concluded that classes had to disappear from the reasoning in which they were present without really letting go of them because mathematics crumbles without a single object to represent an extension.

While wrestling with the fake objects problem, Russell saw parallels between problems arising when classes are treated as objects and those arising when descriptions, 'like the present king of France,' are treated as names. So, he reasoned that since, "class" cannot be a primitive idea. It must be defined along the same lines as the definition of descriptions, be given a definition that assigns a meaning to propositions in which words or symbols apparently representing classes figure, but eliminates all mention of classes from the proper analysis of such propositions. The symbols for classes would then be like descriptions, logical fictions, mere conveniences, not representing objects called classes. Russell believed that his means of drawing objects out of descriptions provided a practical model of how to make non-entities function as entities without incurring contradictory results.

Early in his search for ways to "evade" the contradiction, Russell thought that the answer would be found by "inventing" a hierarchy of classes according to which the first type is composed of classes made up entirely of particulars, the second of classes whose members are classes of the first, the third of classes whose members are classes of the second, etc. The types would be mutually exclusive, making the notion of a class being a member of itself meaningless. This hierarchy was to perform the single, but essential, service of justifying one in not engaging in reasoning that leads to contradictory conclusions, the justification being that what appear to be propositions are actually nonsense.

Russell believed that no solution to the contradictions was technically possible without the theory of types, but realized it was not “the key to the whole mystery.” *After all, it was but an ad hoc effort to restore the hierarchical structure established by the differences between dependent and independent meanings that protects against invalid inference and is broken by axioms of extensionality*.

So, how do I interpret the statements I said I was going to interpret?

The first statement concerned the set-theoretical paradoxes showing that Husserl’s contemporaries did not yet have the genuine concept of set.

Those paradoxes were derived using a concept of set that allows one to form the expression ‘a set may be a member of itself,’ which Husserl judged to be *widersinnig*. He would derive set theory analytically from the a priori concept of set for which a set is a kind of whole subject to the rules governing wholes and parts stipulating that a whole cannot be its own part and that the notion of a set that contains itself as a member is untenable.

The second statement says that if one is clear about meaning, one readily sees the contradiction involved in the set-theoretical paradoxes.

It follows from the above that, if one is clear about the meaning of the genuine concepts of “set,” “member,”“whole” and “part,” one sees that talk of sets being members of themselves is *widersinnig*. For Husserl, being clear about meaning involved recognizing inviolable differences between the dependent and independent meanings that form the necessary basis for discovering the essential categories of meaning grounding the laws of meaning that provide logic with possible coherent, meaningful meaning forms whose truth or falsehood, reference to objects, *Widersinnigkeit* or lack thereof, is determined by logical laws. According to his theory about the differences between logical laws and laws of meaning, something that violates logical laws can genuinely have a coherent meaning and be determined to be true or false, but since it is *widersinnig*, no object can correspond to the meaning. So the logical construction "set of all sets which do not contain themselves as parts" cannot be about something existing any more that the expression "the present emperor of France" can.

For him, being clear about meaning also involved recognizing that sets have an entirely different meaning in the subject-predicate propositions of the first level of logic, where they function as dependent forms, than in set theory on the second level, where they function as higher order objects and truth is the analysis of essences and concepts.

In comparison, Frege reasoned on the first level, which obliged him to treat sets and numbers as objects. He treated numbers as independent objects that must always be conceived substantivally and not as a dependent attributes. He mixed statements in which number properties occur as dependent forms and statements in which numbers are the objects. He mixed the first-level subject-predicate proposition 'Jupiter has four moons' with what Husserl considered to be the second-level arithmetical statement that 2+2=4 and invented a law that would allow him to treat what he recognized as dependent meanings as independent meanings.

I interpret the third statement about the solution to the set-theoretical paradoxes lying in demonstrating the shift of meaning that makes one not immediately aware of the contradiction and unable to indicate wherein it lies as concerning the fundamental distinction between independent and dependent meanings concealed behind inconspicuous grammatical distinctions.

Husserl and Frege agreed about the “fatal tendency” of language to cover up essential differences between concepts and objects and allow a concept word to be transformed into a proper name and figure a place unsuited to it. By an unavoidable “awkwardness of language,” “a kind of necessity of language,” one mentions an object, when one intends a concept. On his copy of Frege’s “On Concept and Object,” Husserl marked the sentence, “Language has means of presenting now one, now another, part of the thought as the subject.” He underlined the word ‘language.’

Frege thought that the word ‘the’ in an expression like ‘the number 1’ sufficed to class the number as an object and that the fact that numbers also appear attributively could be “got around.”He concluded that language’s propensity to undermine the reliability of thinking by forming proper names to which no objects correspond had “dealt the death blow” to his set theory.

Such shifts of meaning allow the pseudo-objects and type ambiguities to creep into reasoning unnoticed that Russell struggled to avoid. As he warned, when two words have different types of meanings, the relations of those words to what they stand for are also of different types and the failure to realize this is “a very potent source of error and confusion in philosophy.” Indeed, if, as Frege stressed, concept words and proper names must occupy essentially different places, if there is a radical difference between dependent and independent meanings, which is such that an object can never stand for a concept or concept for an object, then basic rules of inference like the principle of substitutivity of identicals and existential generalization will fail if the difference is not respected. Blurring distinctions between talk of sets on different levels flattens logical structure and smoothes the way for things to come into places not intended for them. Meaning categories are violated failures of inference result.

In conclusion, I wish to emphasize that Husserl did not say that set theory was false. He considered it a mathematical discipline of the second level of the purely logical sphere, a theory proceeding from purely logical concepts and axioms grounded in purely logical categories like those discovered by the essential distinction between dependent and independent meanings. He concluded that faulty reasoning about a faulty concept of set had led to the set-theoretical paradoxes. Whereas Russell strove to invent ways to evade the contradictions, Husserl advocated starting over and deriving set theory from a non-contradictory concept of set and element, or more universally of whole and part, without resorting to an axiom of extensionality. Russell's struggles illustrate what Husserl meant when he said that extensions generate contradictions requiring every kind of artful device to make them safe for use in mathematical reasoning.

I see nothing paradoxical or mysterious about the contradictions derivable in Frege's logical system. They are just cheap contradictions generated by a contradictory theory of meaning. According to Husserl's theories about the inviolability of laws governing dependent and independent meanings, Russell's contradiction is just saying that the set X of x's is not a member of what it is a set of; what is predicated of an object is of a different logical type from the object itself; a concept is not an object; what is dependent is not independent…. In short, logic is doing what logic is supposed to do.